Reduction of overdamped stochastic processes in the coherent state representation

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The adiabatic elimination of fast variables from overdamped stochastic processes by functional integration is demonstrated. Adiabatic elimination entails the evaluation of a reduced evolution operator from the full evolution equation. For Fokker-Planck processes, the reduced evolution operator may be expressed as a ground state expectation, and it is shown how this is represented as a coherent state path integral. The elimination is then achieved by functionally integrating out all reference to the fast variables. The end result is a decoupling of the full evolution equation into separate equations for the fast and slow variables. The method is demonstrated for Brownian motion and for a system with multiplicative colored noise.

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I. INTRODUCTION

Stochastic systems may be described in terms of a set of randomly fluctuating dynamical variables or degrees of freedom. The fluctuations of these variables with time may be completely independent of history (Markov) or they may have some form of memory (non-Markov). Of the two, Markov processes are considerably easier to handle, but exact solutions are nevertheless rarely possible. One useful simplification arises when a subset of the variables describing a stochastic process fluctuate on time scales much shorter that those of the remaining variables. We call such variables "fast" and "slow," respectively. When this categorization in terms of fast and slow variables is possible, we may in fact eliminate the fast variables from the problem altogether, resulting in a contracted evolution equation in terms of just the slow variables alone [1]. In this reduced description, the influence of the fast variables on the slow ones is felt through parameters appearing in the evolution equation for the slow variables. Such elimination procedures have been used in a number of applications, including laser theory [2] and the study of self-organizing systems [3].

As a concrete example, consider Brownian motion in a potential [4]. This is described by a Fokker-Planck equation (the Kramers-Klein equation) in the particle's velocity and position variables. In the overdamped limit, the friction coefficient is high: the particle's velocity is rapidly relaxing and fluctuates on a time scale much shorter than that of the variations in its position. In this limit, the velocity fluctuations may be characterized approximately by a steady-state distribution function that is independent of the fluctuations in the particle position (the adiabatic approximation). Considering the projection of the joint distribution function onto a state where the velocity distribution is always in equilibrium gives a reduced Fokker-Planck equation in terms of just the position variable. The latter equation is often called the Smoluchowski equation. A number of systematic methods for performing the reduction have been developed, including projection techniques [5–7], cumulant expansions [8,9], and a Chapman-Enskog procedure [10]. These techniques all yield the Smoluchowski equation plus corrections in inverse powers of the friction coefficient. In cases where the adiabatic approximation does not hold, renormalization group ideas have been found to be useful [11,12], but this added complication will not affect us here.

The aforementioned methods for achieving the reduction in the adiabatic approximation do so by algebraic manipulations on the full evolution equation. In this paper, we show that the reduction process can also be expressed in terms of a path integral and solved by quadrature. This path integral is not a standard phase space integral over the position and velocity. Instead, the velocity variable is regarded as independent. The "trajectories" in velocity space are then represented with the aid of coherent states, following which the velocity variable is eliminated through a functional integration. Even though the path integral representing the reduced evolution operator may be Gaussian, an exact evaluation is generally not possible owing to the noncommuting nature of the coefficients that appear in the "action." However, even in this case an approximate evaluation is possible and this in fact reproduces the known cumulant expansion of the reduced evolution operator. Although the analysis presented apparently does no more than use the path integral as a stepping stone to obtain the reduced evolution equation, an important technical advance is achieved in the expression of the problem in a form allowing considerable scope for further investigation and development. The introduction of such functional ideas has proved useful in a number of other areas of physics.

Of course, functional methods are not new in the study of stochastic processes. The Onsager-Machlup [13] theory represents Fokker-Planck processes in terms of path integrals, though not in the same form as used here. This theory has been utilized, for example, in the analysis of escape problems [14]. Functional methods have also been used by Fox [15] in discussions of non-Markov processes. However, functional methods have not yet been applied to the reduction problem as formulated above, despite

the fact that the path integral is a natural form of expression for such problems. They describe very succinctly the interaction of a subsystem with a heat bath [16], and we will find it useful to speak in such terms too. An important step in the analysis is the introduction of boson operators. These have been used in the reduction problem before by a number of authors [6,7,9]. Conventional methods associated with boson operators and the path integral method described here are equivalent, much in the same way as operator and path integral quantization methods in quantum mechanics are equivalent. The present equivalence is manifest in the coherent state representation of the path integral.

In Sec. II we define overdamped processes for a general stochastic system with only additive noise. In Sec. III we show how the reduced evolution operator for such a system is obtained by conventional operator methods. Evaluation of this operator in the adiabatic approximation allows the original evolution equation to be decoupled into equations for the fast and the slow variables. The former are governed by a steady-state equation, the latter by a reduced Fokker-Planck equation. In Sec. IV we demonstrate how the reduced evolution operator may be written as a coherent state path integral and show that the reduction is implemented via a functional integration. We also prove the equivalence of the path integral method to the operator cumulant expansion, which has been used previously in reduction problems. In Sec. V we apply the theory to study Brownian motion. We further study a model system with multiplicative colored noise and show that this too can be handled by the path integral techniques. A formally exact evolution equation for a colored noise system is thereby obtained. We conclude in Sec. VI with a summary of the main achievements.

II. OVERDAMPED STOCHASTIC PROCESSES

In this section we discuss what we mean by an overdamped stochastic process. The concept is clearest when stated in terms of a Langevin equation. For Markov processes, the Langevin equation has a simple structure and is easily transformed into a Fokker-Planck equation in which the inherent diffusive behavior is manifest. Some straightforward manipulation leads to an expression, akin to a vacuum-to-vacuum transition amplitude in quantum theory, which in the high friction limit is amenable to an ordered cumulant expansion and gives the reduced evolution operator. The analysis is readily applied to overdamped Brownian motion in a potential and results in a series expansion for the reduced evolution operator in powers of the inverse friction coefficient. We demonstrate the general principles behind the elimination procedure in Sec. III, following which we introduce the path integral representation in Sec. IV, and then finally study some specific examples in Sec. V. We consider a quite general model on which to base the analysis, with the main constraint being that the noise is assumed to be additive and white. In fact, more general Fokker-Planck equations than the one obtained may be studied [17], but we do not do so here. Later, in Sec. V we will find that multiplicative colored noise can also be handled by the functional method developed.

Describing a Markov process in terms of the N dynamical variables $\{\phi_k\}$, $k = 1,2,\ldots,N$, we write the Langevin equations for a system with free energy F as

$$\dot{\phi}_k = C_k(\phi) - L_k \frac{\partial F}{\partial \phi_k} + \xi_k, \tag{2.1a}$$

$$\langle \xi_k(t)\xi_{k'}(t')\rangle = 2L_k\delta_{kk'}\delta(t-t'), \tag{2.1b}$$

where the overdot denotes a time derivative and the $\xi_k(t)$ are phenomenological Gaussian white noise processes uncorrelated with ϕ_k, with zero mean and strength L_k. Coupling between the dynamical variables is described by $C_k(\phi)$, which satisfies conditions [12,18] such that the equilibrium distribution of the ϕ_k's is

$$p_{\rm eq}(\phi) = e^{-F}. \tag{2.2}$$

Explicit examples of the quantities ϕ_k, $C_k(\phi)$, L_k, and F are given for Brownian motion in Sec. V.

Equation (2.1) may be used to describe the stochastic evolution of many systems, including the aforementioned example of a Brownian particle, the flow of an incompressible Navier-Stokes liquid [18], and the critical dynamics of a Landau-Ginzburg system at a phase boundary [11,12], among others. However, we are not interested in all of these applications. We concentrate instead on those particular systems in which the characteristic time scales over which the ϕ_k relax can be separated into two distinct, widely separated domains. Thus one particular subset of the ϕ_k relax over times scales much shorter than those of the remaining ϕ_k. This defines in a qualitative manner what we mean by an overdamped stochastic system. A more precise definition is given below. We will see that the wider the separation of the two time scales, the more accurate the reduced description of the system dynamics that is obtained.

Assuming that the system described by Eq. (2.1) admits a separation of time scales, we separate the set of dynamical variables $\{\phi_k\}$ into two subsets. One subset contains those variables that are rapidly relaxing in the sense to be defined. These we distinguish with indices in a set $\mathcal{B}$, that is, $k \in \mathcal{B}$. The remainder we distinguish by indices in a set $\mathcal{I}$, that is, $k \in \mathcal{I}$. Thus we have $\{\phi_k\} = \{\phi_{k\in\mathcal{B}}\} + \{\phi_{k\in\mathcal{I}}\}$. Then, expanding the free energy in powers of ϕ_k, we single out the quadratic term

$$F(\phi) = \sum_k \frac{\phi_k^2}{2D_k} + f(\phi). \tag{2.3}$$

In some circumstances it may be more appropriate to write $|\phi_k|^2$ in place of ϕ_k^2. With the free energy expressed in this form, Eq. (2.1) yields the Fokker-Planck equation for the distribution function $p(\phi)$,

$$\frac{\partial}{\partial t}p(\phi) = (L_b + L_i)p(\phi), \tag{2.4a}$$

$$L_b = \sum_{k\in\mathcal{B}} \lambda_k \frac{\partial}{\partial \phi_k}\left(\phi_k + D_k\frac{\partial}{\partial \phi_k}\right), \tag{2.4b}$$

$$L_i = \sum_{k\in\mathcal{I}} \lambda_k \frac{\partial}{\partial\phi_k}\left(\phi_k + D_k\frac{\partial}{\partial\phi_k}\right) + \sum_k \frac{\partial}{\partial\phi_k}\left(L_k\frac{\partial f}{\partial\phi_k} - C_k(\phi)\right), \quad (2.4c)$$

where we have written $L_k = \lambda_k D_k$. The summations over $k \in \mathcal{B}$ and over $k \in \mathcal{I}$ must obviously satisfy

$$\sum_k A(\phi) = \sum_{k\in\mathcal{B}} A(\phi) + \sum_{k\in\mathcal{I}} A(\phi) \quad (2.5)$$

for an arbitrary differential operator $A(\phi)$.

By writing the Fokker-Planck equation in the form above, we have isolated terms describing free Brownian motion proportional to the parameters λ_k. We now define an overdamped stochastic process as one in which

$$\lambda_{k\in\mathcal{B}} \gg \lambda_{k\in\mathcal{I}}, \quad (2.6)$$

i.e., as one in which the coefficients λ_k connected to the "rapidly relaxing" variables have much larger magnitudes than those of the remaining variables. The association of $1/\lambda_k$ with a time scale emerges by considering a free energy with a dominant quadratic term and by neglecting the coupling between the dynamical variables: the mean of ϕ_k is then found to satisfy

$$\langle\phi_k(t)\rangle = \langle\phi_k(0)\rangle e^{-\lambda_k t}, \quad (2.7)$$

which clearly exhibits λ_k as a decay rate, larger λ_k implying shorter characteristic decay times. Of course, the relaxation behavior of the ϕ_k will not be of this form for general processes. However, the inequality (2.6) implies that for the systems of interest the total free energy is dominated by the quadratic term associated with the $\phi_{k\in\mathcal{B}}$'s, i.e., $F \sim \sum_{k\in\mathcal{B}} \phi_k^2/2D_k$, and that the noise strengths L_k for $k \in \mathcal{B}$ are much larger than those for $k \in \mathcal{I}$. Both of these conditions are consistent with a rapidly relaxing behavior for the fast variables. In Brownian motion, λ_k is the friction coefficient and the overdamped limit corresponds to the limit of high friction.

III. REDUCED FOKKER-PLANCK EQUATION

We wish to reduce the Fokker-Planck equation above, Eq. (2.4a), into a contracted form taking advantage of the condition (2.6). Stated differently, we are looking for a means of decoupling the evolution of the slow and fast variables, which are presently coupled in the one full equation. Two distinct strategies are conceivable, depending on the time scales of the processes of most interest. The first is to consider the influence of the slow variables on the fast and to obtain an evolution equation for the latter. Such a strategy is most appropriate when considering processes occurring over very short time scales. The instantaneous values of the slow variables then govern the evolution of the fast variables in a manner to be found. This is akin to the "slaving" principle proposed by Haken [3] in the context of self-organizing systems. In this paper we do not consider processes on short time scales, but instead focus on the second strategy, which is to consider the influence of the fast variables on the slow in the adiabatic approximation. This is most appropriate for processes occurring over very long time scales.

Now, because of condition (2.6) the fast variables may be described by a quasiequilibrium distribution that may depend on the instantaneous values of the slow variables (see the preceding paragraph). We will assume that the slow variables do not influence the dynamics of the fast variables at all, i.e., that any coupling term can be ignored in the evolution equation for the fast variables. Over long time scales, the distribution function of the fast variables is then described by the steady-state equation

$$L_b\rho_{\rm eq}(\phi_{\mathcal{B}}) = 0, \quad (3.1a)$$

so that

$$\rho_{\rm eq}(\phi_{\mathcal{B}}) = \prod_{k\in\mathcal{B}} \frac{1}{\sqrt{2\pi D_k}} e^{-\phi_k^2/2D_k}, \quad (3.1b)$$

where $\phi_{\mathcal{B}}$ denotes the set of ϕ_k with $k \in \mathcal{B}$. This is the adiabatic approximation in the form of most use to us. In this approximation, Eq. (3.1) trivially constitutes one-half of the solution to the desired decoupling of the full Fokker-Planck equation. The actual problem is to find the evolution equation for the slow variables.

The condition that the distribution function for the fast variables is unaffected by the slow variables, whereas the slow variables might be affected by the fast variables, is exactly the condition satisfied by an ideal heat bath [19]. In equilibrium statistical physics, a heat bath is used to impose certain equilibrium properties on the subsystem with which it is in contact. The heat bath is "large" in some sense and is totally unaffected by its coupling to the subsystem. Such an idea is consistent with the requirement made at the end of Sec. II for the free energy (recalling it is an additive quantity) to be dominated by the prospective heat bath $\{\phi_{k\in\mathcal{B}}\}$ in equilibrium. On the other hand, the properties of the subsystem will respond to the coupling with the heat bath. In a nonequilibrium situation, this coupling results in dissipative forces causing the distribution function of the subsystem to relax towards a steady state determined by the heat bath. Thus the approximation we have used above to write Eq. (3.1) is effectively to regard the fast variables as constituting a heat bath in which the subsystem described by the slow variables is immersed. This is a good approximation only provided condition (2.6) holds.

With the distribution of the fast variables given by $\rho_{\rm eq}(\phi_{\mathcal{B}})$, the total distribution function has the form

$$p(\phi) = \rho_{\rm eq}(\phi_{\mathcal{B}})\sigma(\phi_{\mathcal{I}}), \quad (3.2)$$

where $\phi_{\mathcal{I}}$ denotes the set of ϕ_k with $k \in \mathcal{I}$. Now the eigenvalue problem for L_b may be solved [7] to give $L_b\psi_{n,k}(\phi_k) = -n_k\psi_{n,k}(\phi_k)$, showing that $\rho_{\rm eq}(\phi_{\mathcal{B}}) \propto \psi_0(\phi_{\mathcal{B}})$, with $\psi_0(\phi_{\mathcal{B}}) = \prod_{k\in\mathcal{B}}\psi_{0,k}$. Thus $\sigma(\phi_{\mathcal{I}})$ is in fact obtained by projecting $p(\phi)$ onto the $n = 0$ eigenstate, or ground state, of L_b. The adjoint eigenvalue problem may also be solved, giving $L_b^\dagger\tilde{\psi}_{n,k}(\phi_k) = -n_k\tilde{\psi}_{n,k}(\phi_k)$,

with $\tilde{\psi}_0(\phi_{\mathcal{B}}) = \prod_{k\in\mathcal{B}} \tilde{\psi}_{0,k}$. Using the similarity transformation

$$\bar{p}(\phi) = e^{-L_b t} p(\phi), \qquad L_i(t) = e^{-L_b t} L_i e^{L_b t}, \tag{3.3}$$

Eq. (2.4a) may be cast in the form

$$\frac{\partial}{\partial t}\bar{p}(\phi) = L_i(t)\bar{p}(\phi), \tag{3.4}$$

which has the solution

$$\bar{p}(\phi) = R(t)\bar{p}_0(\phi), \tag{3.5a}$$

$$R(t) = T\exp\left(\int_0^t dt' L_i(t')\right), \tag{3.5b}$$

with $\bar{p}(\phi) = \bar{p}_0(\phi)$ at $t = 0$ and T being the time-ordering operator. From this, some algebraic manipulation [9] shows that

$$\sigma(\phi_{\mathcal{I}}) = R_{\rm red}(t)\sigma_0(\phi_{\mathcal{I}}), \tag{3.6a}$$

$$R_{\rm red}(t) = \int \prod_{k\in\mathcal{B}} d\phi_k \tilde{\psi}_0(\phi_{\mathcal{B}}) R(t) \psi_0(\phi_{\mathcal{B}}), \tag{3.6b}$$

where $\sigma(\phi_{\mathcal{I}}) = \sigma_0(\phi_{\mathcal{I}})$ at $t = 0$. Thus, just as the state $\sigma(\phi_{\mathcal{I}})$ is the ground state projection of $p(\phi)$, Eq. (3.6b) shows that the operator $R_{\rm red}(t)$ is the ground state projection (expectation) of $R(t)$, where we note that $R(t)$ expresses the time evolution of $p(\phi)$ in the "interaction" representation defined by Eq. (3.3). In quantum mechanical terms, Eq. (3.6b) is akin to a vacuum-to-vacuum amplitude.

The expression for $R_{\rm red}(t)$ given in Eq. (3.6b) may be simplified using an ordered cumulant expansion of the expectation. This technique has been discussed in the context of the reduction problem by a number of authors [6,8,9]. Using such an ordered cumulant expansion, $R_{\rm red}(t)$ may be rewritten in the form

$$R_{\rm red}(t) = T\exp\left(\int_0^t dt' L_{\rm eff}(t')\right). \tag{3.7}$$

Now reversing the procedure used to obtain Eq. (3.5) from Eq. (3.4), we recover the differential equation

$$\frac{\partial}{\partial t}\sigma(\phi_{\mathcal{I}}) = L_{\rm eff}(t)\sigma(\phi_{\mathcal{I}}). \tag{3.8}$$

This is the differential equation satisfied by $\sigma(\phi_{\mathcal{I}})$ if the fast variables are in equilibrium to start off with at $t = 0$. If this is not the case, however, then we must further ensure that this "solution" also satisfies the long time scale condition discussed at the start of this section, since we have not made any special arrangements to include processes occurring on short time scales. In other words, we must ensure that the differential operator governing the evolution of $\sigma(\phi_{\mathcal{I}})$ describes only processes on a time scale much longer than the characteristic decay times of the fast variables. Thus we must replace $L_{\rm eff}(t)$ in Eq. (3.8) by

$$L_{\rm red}(t) = \lim_{t\gg 1/\lambda_{k\in\mathcal{B}}} L_{\rm eff}(t), \tag{3.9}$$

thereby ignoring any apparent processes occurring over very short time scales (transients). This solution is independent of the distribution of the fast variables at time $t = 0$, which may be taken to be in nonequilibrium if desired: over time scales $t \gg 1/\lambda_{k\in\mathcal{B}}$, this distribution will reach the steady-state form $\rho_{\rm eq}(\phi_{\mathcal{B}})$ and the above analysis becomes valid. The aforementioned transients may be avoided by switching on the heat bath coupling adiabatically from the infinite past rather than suddenly (and artificially) at $t = 0$ as done above.

Equations (3.1), (3.8), and (3.9) represent a solution to our original problem, namely, that of decoupling the original Fokker-Planck equation Eq. (2.4a) by making use of the separation of time scales condition Eq. (2.6). Equations (3.6)–(3.9) for the reduced system are analogous to Eqs. (3.4) and (3.5) for the full system. These reduced equations are obtained essentially by a projection of the original problem onto the ground state of L_b, expressing the fact that we have regarded the fast variables as constituting a heat bath. This decoupling was achieved originally by Kramers [20] and others for the problem of overdamped Brownian motion in a potential and has since been discussed for a number of other systems involving slowly and rapidly fluctuating dynamical variables [6,7,21]. Our review of the problem in Secs. I and II has been expressed slightly differently than is usually done, highlighting the interpretation of the reduction process in terms of heat baths. Having set up the problem in some detail, we show in the following sections that the reduction process may be formulated very elegantly in the framework of functional integration, which is a very natural expression of the heat bath concept.

IV. REDUCTION IN THE COHERENT STATE REPRESENTATION

Before proceeding to the path integral representation of the reduction problem, we transform the full Fokker-Planck equation (2.4a) into a more useful form. This entails writing the Fokker-Planck equation as a Schrödinger equation in imaginary time, with a "Hamiltonian" expressed in terms of creation and annihilation operators. The evolution operator associated with this Schrödinger equation may be written in path integral form in the coherent state representation, which is the most convenient representation for subsequent manipulations. The reduction problem is then simply a matter of performing the relevant functional integration. However, this functional integral cannot generally be performed exactly and a perturbative evaluation may be necessary. This is demonstrated by way of an example in Sec. V B. The main result of this paper is contained in this section, that is, the expression of the reduction (or decoupling) problem in terms of a path integral. The reason why a coherent state path integral is used as opposed to one of the

more traditional variety becomes clear when applying the method.

As the first step, we use

$$\mathcal{U} = \prod_{k\in\mathcal{B}} e^{\phi_k^2/4D_k} \tag{4.1}$$

to perform the similarity transformation

$$P(\phi) = \mathcal{U}p(\phi), \tag{4.2a}$$

$$H = H_b + H_i = -\mathcal{U}(L_b + L_i)\mathcal{U}^{-1} \tag{4.2b}$$

on Eq. (2.4a) to give the imaginary time Schrödinger equation

$$-\frac{\partial}{\partial t}P(\phi) = HP(\phi). \tag{4.3}$$

Likewise transforming the eigenstates of L_b and $L_b^\dagger$ gives $\Psi_{n,k}(\phi_k) = \mathcal{U}\psi_{n,k}(\phi_k) = \mathcal{U}^{-1}\tilde{\psi}_{n,k}(\phi_k)$ for $k \in \mathcal{B}$, which are eigenstates of the Hamiltonian H_b since $H_b\Psi_{n,k}(\phi_k) = n_k\Psi_{n,k}(\phi_k)$. The $\Psi_{n,k}(\phi_k)$ are in fact harmonic oscillator states, as may be seen by writing the eigenvalue equation explicitly in differential form. The $\frac{1}{2}$ that usually appears in the quantum mechanical version of the eigenvalue equation is here incorporated into H_b and does not appear explicitly.

A little algebra shows that H_b may be written as

$$H_b = \sum_{k\in\mathcal{B}} \lambda_k B_k^\dagger B_k, \tag{4.4}$$

where

$$B_k^\dagger = \frac{\phi_k}{2\sqrt{D_k}} - \sqrt{D_k}\frac{\partial}{\partial\phi_k}, \tag{4.5a}$$

$$B_k = \frac{\phi_k}{2\sqrt{D_k}} + \sqrt{D_k}\frac{\partial}{\partial\phi_k} \tag{4.5b}$$

are creation and annihilation operators for the states $\Psi_{n,k}(\phi_k)$,

$$[B_k, B_{k'}^\dagger] = \delta_{kk'}, \tag{4.6a}$$

$$B_k^\dagger\Psi_{n,k}(\phi_k) = \sqrt{n+1}\,\Psi_{n+1,k}(\phi_k), \tag{4.6b}$$

$$B_k\Psi_{n,k}(\phi_k) = \sqrt{n}\,\Psi_{n-1,k}(\phi_k). \tag{4.6c}$$

From Eq. (4.5) we may write ϕ_k and $\partial/\partial\phi_k$ for $k \in \mathcal{B}$ in terms of B_k and $B_k^\dagger$. Thus the interaction H_i may also be written in terms of B_k and $B_k^\dagger$.

We now introduce coherent states [22–24] into the picture. Coherent states are eigenstates $b_k(\phi_k)$ of the annihilation operator

$$B_k b_k(\phi_k) = \beta_k b_k(\phi_k), \qquad B_k^\dagger\tilde{b}_k(\phi_k) = \beta_k^*\tilde{b}_k(\phi_k), \tag{4.7}$$

where $\tilde{b}_k(\phi_k)$ is the adjoint of $b_k(\phi_k)$ and the eigenvalue β_k is defined on the set of complex numbers. In terms of the ground state $\Psi_{0,k}(\phi_k)$, we have

$$b_k(\phi_k) = e^{\beta_k B_k}\Psi_{0,k}(\phi_k). \tag{4.8}$$

The set of states $b_k(\phi_k)$ is overcomplete in that there is one state too many to form a linearly independent set. However, the set possesses a resolution of identity, albeit nonunique, and so may be used for the purposes of constructing eigenfunction expansions.

The solution to Eq. (4.3) may be written formally

$$P(\phi) = e^{-Ht}P_0(\phi), \tag{4.9}$$

where $P(\phi) = P_0(\phi)$ at $t = 0$. Using the identity

$$e^{-(H_b+H_i)t} = e^{-H_b t}T\exp\left(-\int_0^t dt' H_i(t')\right) \tag{4.10}$$

with $H_i(t) = e^{H_b t}H_i e^{-H_b t}$, together with the fact that

$$T\exp\left(-\int_0^t dt' H_i(t')\right) = \mathcal{U}R(t)\mathcal{U}^{-1} \tag{4.11}$$

leads to the important relation

$$\int\prod_{k\in\mathcal{B}} d\phi_k\Psi_0(\phi_\mathcal{B})e^{-Ht}\Psi_0(\phi_\mathcal{B}) = \int\prod_{k\in\mathcal{B}} d\phi_k\tilde{\psi}_0(\phi_\mathcal{B})R(t)\psi_0(\phi_\mathcal{B}), \tag{4.12}$$

where $\Psi_0(\phi_\mathcal{B}) = \prod_{k\in\mathcal{B}}\Psi_{0,k}(\phi_k)$. Representing $\Psi_0(\phi_\mathcal{B})$ in the abstract form $|0\rangle$ and recalling Eq. (3.6b), the above may be rewritten much more transparently as

$$R_{\rm red}(t) = \langle 0|e^{-Ht}|0\rangle. \tag{4.13}$$

This simple result tells us directly that the reduced operator $R_{\rm red}(t)$ is just the ground state projection of the full evolution operator as defined in Eq. (4.9).

Of course, we could have reached the same conclusion without recourse to the various transformations and manipulations above. However, the advantage of the representation of $R_{\rm red}(t)$ in Eq. (4.13) in terms of harmonic oscillator states and creation and annihilation operators is that the expectation can be rewritten directly as a coherent state path integral. The traditional form of the path integral [25] follows by writing the transition amplitude of e^{-Ht} in a basis of position eigenstates, e.g., $\langle q_2|e^{-Ht}|q_1\rangle$. This latter quantity may be expressed as a path integral over all paths $q(t)$ with fixed end points q_1 and q_2, i.e., with boundary conditions $q(0) = q_1$ and $q(t) = q_2$. In Eq. (4.13) we have written $R_{\rm red}(t)$ as a transition amplitude not using position eigenstates but instead using coherent states. The general form of such a transition amplitude is $\langle b''|e^{-Ht}|b'\rangle$ for coherent states $|b'\rangle$ and $|b''\rangle$ which are products of the individual eigenstates of B_k, $k \in \mathcal{B}$. This transition amplitude may be written as a path integral on the space of complex functions $\beta_k(t)$ satisfying the constraints $\beta_k(0) = b_k'$ and $\beta_k(t) = b_k''$. For further details and applications of coherent state path integrals, the reader is referred to textbooks [23,24].

The actual transition amplitude in Eq. (4.13) involves two ground, or vacuum, states. Thus it is a vacuum-to-vacuum transition amplitude and may be written as a path integral over complex functions satisfying the periodic boundary conditions $\beta_k(0) = \beta_k(t) = 0$. Details of how to do so are described in standard textbooks [23,24] and we do not repeat the algebra here. The final result may be written in the form

$$R_{\rm red}(t) = \frac{1}{N}\oint \mathcal{D}(\beta^*\beta) \times \exp\left[-\int_0^t d\tau \left(\sum_k \beta_k^* \partial_\tau \beta_k + H(\beta^*,\beta)\right)\right], \tag{4.14a}$$

where

$$\oint \mathcal{D}(\beta^*\beta) \equiv \prod_{k\in B}\int_{\beta_k(0)=0}^{\beta_k(t)=0} \mathcal{D}\beta_k^*\mathcal{D}\beta_k \tag{4.14b}$$

and ∂_τ denotes a time derivative. The normalization is

$$N = \oint \mathcal{D}(\beta^*\beta) \times \exp\left[-\int_0^t d\tau \left(\sum_k \beta_k^* \partial_\tau \beta_k + H_b(\beta^*,\beta)\right)\right]. \tag{4.15}$$

which is analogous to Eq. (4.14a) except with H replaced by H_b. These Hamiltonians are given in the coherent state representation simply by

$$H_b(\beta^*,\beta) = \sum_k \lambda_k \beta_k^* \beta_k, \tag{4.16a}$$

$$H(\beta^*,\beta) = H_b(\beta^*,\beta) + H_i(\beta^*,\beta). \tag{4.16b}$$

Specific examples are given in Sec. V.

Since we are in imaginary time, the path integral in Eq. (4.14a) may be given a rigorous probabilistic definition [26]. This allows us to rewrite the path integral for $R_{\rm red}(t)$ in the alternative but equivalent notation as

$$R_{\rm red}(t) = \left\langle \exp\left(-\int_0^t d\tau\, H_i(\beta^*,\beta)\right)\right\rangle. \tag{4.17}$$

The probabilistic definition of the imaginary time path integral allows the angular brackets $\langle\,\rangle$ to be interpreted in two equivalent ways. First, they may be interpreted as a functional average that may be written as a functional integral using the functional measure known as the Ornstein-Uhlenbeck measure (a generalization of the Weiner measure) [26]. This leads back directly to Eq. (4.14a). Alternatively, they may be interpreted as an operator average [24]

$$\left\langle \exp\left(-\int_0^t d\tau\, H_i(\beta^*,\beta)\right)\right\rangle = \left\langle 0\left| T\exp\left(-\int_0^t dt'\, H_i(t')\right)\right| 0\right\rangle, \tag{4.18}$$

which is effectively the representation we started out with to derive the path integral in the first place. This somewhat tautological argument underlines the equivalence of the path integral representation to the standard representation of $R_{\rm red}(t)$, showing that they really are one and the same fundamental object. Equation (4.17) is the analogue of what in quantum theory is sometimes called the Feynman-Kac formula.

The preceding discussion contains the essential result of this paper, that is, that the reduced evolution operator $R_{\rm red}(t)$, using which the full Fokker-Planck equation may be decoupled into separate equations for the fast and slow variables, may be represented as a path integral. The "integrand" in this path integral contains the full dynamics of the problem, as is evident from the presence of the total Hamiltonian $H(\beta^*,\beta)$. The reduced dynamics is then obtained by integrating out the dynamics associated with the fast variables, which we have done using boson coherent states $b_k(\phi_k)$ with $k \in \mathcal{B}$. The final answer for $R_{\rm red}(t)$ contains reference only to the slow variables. It turns out that for all but the simplest cases, the path integral cannot be evaluated exactly and we must resort to approximate methods. One such method is the ordered cumulant expansion of the path integral. Due to Eq. (4.18) this in fact leads to exactly the same results mentioned in Sec. III. The fact that we are only rederiving known results using a seemingly more complicated formalism is not a problem. The important point is that the introduction of functional ideas lays the foundation for further development and application of the reduction method for overdamped stochastic processes.

V. APPLICATIONS

A. Free Brownian motion

The example of free Brownian motion is one instance where an exact evaluation of the path integral is possible. The Langevin equations are

$$\dot r_k = v_k, \qquad \dot v_k = -\lambda_k v_k + \xi_k, \tag{5.1a}$$

$$\langle \xi_k(t)\xi_{k'}(t')\rangle = 2\lambda_k D\delta_{kk'}\delta(t-t'), \tag{5.1b}$$

where $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity of the particle, with r_k $(k = 1,\ldots,N)$ being the components of $\mathbf{r}$ and likewise for $\mathbf{v}$, etc. Also, the λ_k are phenomenological friction coefficients along directions with index k and the Einstein relation gives $D = k_BT/m$. Equation (5.1) is equivalent to Eq. (2.1) with the identification

$$\vec\phi = \begin{pmatrix}\mathbf{r}\\ \mathbf{v}\end{pmatrix}, \qquad \vec L = \begin{pmatrix}0\\ \lambda_k D\end{pmatrix}, \tag{5.2a}$$

$$\vec C(\phi) = \begin{pmatrix}\mathbf{v}\\ 0\end{pmatrix}, \tag{5.2b}$$

$$F = \sum_k \left\{\frac{v_k^2}{2D} + \tfrac{1}{2}\ln(2\pi D)\right\}. \tag{5.2c}$$

The form of the free energy F follows from the equilibrium distribution for the $\mathbf{v}$ fluctuations Eq. (3.1b), which for free Brownian motion is given by the Maxwell distribution

$$\rho_{\rm eq}(\mathbf{v}) = \frac{1}{(2\pi D)^{N/2}} \exp\left(-\sum_k m v_k^2/2k_BT\right). \quad (5.3)$$

The Fokker-Planck equation corresponding to Eq. (5.1) is

$$\frac{\partial}{\partial t} p(\mathbf{r},\mathbf{v}) = (L_b + L_i) p(\mathbf{r},\mathbf{v}), \quad (5.4a)$$

$$L_b = \sum_k \lambda_k \frac{\partial}{\partial v_k}\left(v_k + D\frac{\partial}{\partial v_k}\right), \quad (5.4b)$$

$$L_i = -\sum_k v_k \frac{\partial}{\partial r_k}. \quad (5.4c)$$

This describes the unconstrained diffusion of a Brownian particle in a $2N$-dimensional phase space. We are interested in the overdamped limit, when λ_k becomes very large.

Performing the similarity transformation described in Sec. IV, we obtain an imaginary time Schrödinger equation with the Hamiltonian

$$H = \sum_k \lambda_k B_k^\dagger B_k + \sum_k \sqrt{D}\frac{\partial}{\partial r_k}(B_k + B_k^\dagger). \quad (5.5)$$

The reduced evolution, operator is then

$$R_{\rm red}(t) = \frac{1}{N}\oint \mathcal{D}(\beta^*\beta) \exp\left(-\int_0^t \sum_k \left\{\beta_k^* \Omega_k \beta_k + \sqrt{D}\frac{\partial}{\partial r_k}(\beta_k + \beta_k^*)\right\}\right), \quad (5.6)$$

where $\Omega_k = \partial_\tau + \lambda_k$. To evaluate this, we use the following result valid for ordinary bilinear (Gaussian) coherent state path integrals:

$$\frac{1}{N}\oint \mathcal{D}(\beta^*\beta) \exp\left(-\int_0^t \sum_k \{\beta_k^*\Omega_k\beta_k - J_k^*\beta_k - J_k\beta_k^*\}\right)$$

$$= \exp\left(\int_0^t dt_1 \int_0^t dt_2 \sum_k J_k^*(t_1) S_k(t_1 - t_2) J_k(t_2)\right), \quad (5.7a)$$

where

$$S_k(t_1 - t_2) = \tfrac{1}{2} e^{-\lambda_k |t_1 - t_2|} \quad (5.7b)$$

and the $J_k(t)$ are arbitrary functions. Equation (5.7) is effectively a "zero temperature" boson coherent state path integral. Such path integrals may be evaluated by discretization, as shown, for example, by Negele and Orland [24].

Applying Eq. (5.7) to Eq. (5.6) we readily obtain

$$R_{\rm red}(t) = \exp\left(\int_0^t dt' L_{\rm eff}(t')\right), \quad (5.8a)$$

$$L_{\rm eff}(t) = \sum_k \frac{1}{\lambda_k} D(1 - e^{-\lambda_k t}) \frac{\partial^2}{\partial r_k^2} \quad (5.8b)$$

and the time ordering is not necessary. Now using Eq. (3.9) gives

$$L_{\rm red} = \sum_k \frac{1}{\lambda_k} D \frac{\partial^2}{\partial r_k^2}, \quad (5.9)$$

whence the reduced evolution equation is

$$\frac{\partial}{\partial t}\sigma(\mathbf{r}) = \sum_k \frac{1}{\lambda_k} D \frac{\partial^2}{\partial r_k^2}\sigma(\mathbf{r}). \quad (5.10)$$

This equation is exact within the adiabatic approximation and is known as the Einstein equation. It is the overdamped limit of the original Fokker-Planck equation (5.4). Stated differently, we have decoupled the dynamics of the position and velocity variables so that the total probability distribution has the approximate form $p(\mathbf{r},\mathbf{v}) \approx \sigma(\mathbf{r})\rho_{\rm eq}(\mathbf{v})$. Note that because the quadratic path integral for free Brownian motion can be evaluated exactly, the above method is not the only method of evaluation that works. The path integral may also be described in terms of ordinary "trajectories" in velocity space rather than with coherent states.

B. Brownian motion in potential

Brownian motion in a potential is more difficult to handle than free Brownian motion because the path integral cannot be integrated exactly despite being of Gaussian form. This is due to the noncommuting nature of the coefficients appearing in the "action" (i.e., the exponent of the integrand). The Langevin equations this time are

$$\dot{r}_k = v_k, \qquad \dot{v}_k = -\lambda_k v_k - \frac{1}{m}\frac{\partial U}{\partial r_k} + \xi_k, \quad (5.11a)$$

$$\langle \xi_k(t)\xi_{k'}(t')\rangle = 2\lambda_k D \delta_{kk'}\delta(t - t'), \quad (5.11b)$$

where the potential $U(\mathbf{r})$ has been introduced into the velocity equation. This is equivalent to the model introduced in Sec. II through the identification

$$\vec{C}(\phi) = \begin{pmatrix} \mathbf{v} \\ -\frac{1}{m}\partial_{r_k} U \end{pmatrix}, \quad (5.12a)$$

$$F = F_0 + \frac{U(\mathbf{r})}{k_BT} + \sum_k \frac{m v_k^2}{2k_BT} \quad (5.12b)$$

for F_0 a constant and with $\vec{\phi}$ and $\vec{L}$ as in Eq. (5.2a). The free energy emerges from the Maxwell-Boltzmann distribution. The Fokker-Planck equation is

$$\frac{\partial}{\partial t}p(\mathbf{r},\mathbf{v}) = (L_b + L_i)p(\mathbf{r},\mathbf{v}), \tag{5.13a}$$

$$L_b = \sum_k \lambda_k \frac{\partial}{\partial v_k}\left(v_k + D\frac{\partial}{\partial v_k}\right), \tag{5.13b}$$

$$L_i = \sum_k \left(\frac{1}{m}\frac{\partial U}{\partial r_k}\frac{\partial}{\partial v_k} - v_k\frac{\partial}{\partial r_k}\right). \tag{5.13c}$$

Comparing this to the Fokker-Planck equation for free Brownian motion shows that the only difference lies in the interaction term L_i.

A similarity transformation now gives an imaginary time Schrödinger equation with the Hamiltonian

$$H = \sum_k \lambda_k B_k^\dagger B_k + \sum_k (\overline{A}_k B_k + A_k B_k^\dagger), \tag{5.14a}$$

$$\overline{A}_k = \sqrt{D}\frac{\partial}{\partial r_k}, \qquad A_k = \frac{1}{m\sqrt{D}}\frac{\partial U}{\partial r_k} + \sqrt{D}\frac{\partial}{\partial r_k}. \tag{5.14b}$$

By contrast to Hamiltonians typically appearing in quantum mechanical problems (e.g., in the polaron problem [25]) the coefficients $\overline{A}_k$ and A_k do not commute in general:

$$[A_k, \overline{A}_{k'}] = -\frac{1}{m}\frac{\partial^2 U}{\partial r_k \partial r_{k'}}. \tag{5.15}$$

This distinguishes the stochastic problem from ordinary quantum mechanical problems and means that the corresponding path integral cannot be evaluated exactly. We write the reduced evolution operator in the form of a functional average

$$R_{\text{red}}(t) = \left\langle \exp\left(-g\int_0^t dt' \sum_k (\overline{A}_k\beta_k + A_k\beta_k^*)\right)\right\rangle, \tag{5.16}$$

where we have introduced a parameter g for convenience, which we will later set to unity. This path integral may be evaluated approximately in an ordered cumulant expansion, which in fact leads to the same results obtained by operator methods. As opposed to the path integral for free Brownian motion, expressing the path integral in terms of ordinary trajectories leads to difficulties in evaluation due to the noncommuting nature of $\overline{A}_k$ and A_k. This is the reason behind the expression of the path integral in coherent state form.

To derive the ordered cumulant expansion for the path integral, we write

$$R_{\text{red}}(t) = T\exp\left(\int_0^t dt' L_{\text{eff}}(t')\right) \tag{5.17}$$

and expand $L_{\text{eff}}(t)$ in powers of g,

$$L_{\text{eff}}(t) = -gL_1(t) - g^2L_2(t) - g^3L_3(t) - \cdots. \tag{5.18}$$

Further expanding the exponentials in Eqs. (5.17) and (5.16) and equating powers of g gives the van Kampen rules for evaluating the terms in Eq. (5.18) [27,28]

$$L_1(t_1) = \langle 1\rangle, \tag{5.19a}$$

$$L_2(t_1) = -\int_0^{t_1} dt_2\left(\langle 12\rangle - \langle 1\rangle\langle 2\rangle\right), \tag{5.19b}$$

$$L_3(t_1) = \int_0^{t_1} dt_2 \int_0^{t_2} dt_3\ (\langle 123\rangle - \langle 12\rangle\langle 3\rangle - \langle 1\rangle\langle 23\rangle - \langle 13\rangle\langle 2\rangle + \langle 1\rangle\langle 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle\langle 2\rangle), \tag{5.19c}$$

and so on, where, e.g., $\langle 12\rangle = \langle 0|H_i(t_1)H_i(t_2)|0\rangle$ are moments of the interaction

$$H_i(t) = e^{H_b t}H_i e^{-H_b t} = \sum_k (\overline{A}_k B_k e^{-\lambda_k t} + A_k B_k^\dagger e^{\lambda_k t}). \tag{5.20}$$

Recalling the properties of the creation and annihilation operators Eq. (4.6) the lower-ordered cumulants $L_1(t)$, $L_2(t)$,..., may be evaluated straightforwardly. Evaluation of higher-ordered cumulants is aided by diagrammatic analysis, as shown, for example, by Steiger and Fox [9].

The result of this evaluation up to order g^2 is

$$L_{\text{eff}}(t) = \sum_k \frac{1}{\lambda_k}(1 - e^{-\lambda_k t})\frac{\partial}{\partial r_k}\left(\frac{1}{m}\frac{\partial U}{\partial r_k} + D\frac{\partial}{\partial r_k}\right), \tag{5.21}$$

whence the reduced Fokker-Planck equation is

$$\frac{\partial}{\partial t}\sigma(\mathbf{v}) = \sum_k \frac{1}{\lambda_k}\frac{\partial}{\partial r_k}\left(\frac{1}{m}\frac{\partial U}{\partial r_k} + D\frac{\partial}{\partial r_k}\right)\sigma(\mathbf{v}), \tag{5.22}$$

also known as the Smoluchowski equation. As opposed to the Einstein equation for free Brownian motion, the Smoluchowski equation is not exact. However, it is a good approximation for $\lambda_k \gg 1$ because the expansion for $L_{\text{eff}}(t)$ emerges as an expansion in inverse powers of λ_k [10], though we have not proved this. Thus we have again achieved a decoupling of the original Fokker-Planck equation, but this time with greater approximation than was the case with free Brownian motion. As an added note, the problem with the above cumulant expansion must be mentioned. The difficulty relates to the fact that the expansion has been carried out in an arbitrary parameter g, which has been inserted artificially, whereas the natural expansion parameter is $1/\lambda_k$. This means that the expansion above has an undefined rate of convergence.

C. Colored noise

As a final example of where the reduction procedure may be implemented using path integrals, we consider

a simple system with multiplicative, exponentially correlated noise. The effect of colored noise has been found to be important in a number of physical systems [29,30], but we will do little more here than derive a formal evolution equation using the path integral method developed above. We write the Langevin equations as

$$\dot{\phi}_k = W_k(\phi) + g_k(\phi)\eta_k, \tag{5.23a}$$

$$\langle \eta_k(t)n_{k'}(t')\rangle = M_k e^{-\gamma_k|t-t'|}\delta_{kk'}, \tag{5.23b}$$

where $W_k(\phi)$ and $g_k(\phi)$ may be nonlinear functions of the ϕ_k, and $\eta_k(t)$ represents the noise which we have assumed to be exponentially correlated. Equation (5.23) may be written as the Fokker-Planck equation

$$\frac{\partial}{\partial t}p(\phi,\eta) = (L_a + L_b + L_i)p(\phi,\eta), \tag{5.24a}$$

$$L_a = -\sum_k \frac{\partial}{\partial\phi_k}W_k(\phi), \tag{5.24b}$$

$$L_b = \sum_k \gamma_k \frac{\partial}{\partial\eta_k}\left(\eta_k + M_k\frac{\partial}{\partial\eta_k}\right), \tag{5.24c}$$

$$L_i = -\sum_k \frac{\partial}{\partial\phi_k}g_k(\phi)\eta_k \tag{5.24d}$$

for the distribution $p(\phi,\eta)$. We desire to find the evolution equation for the distribution function of just ϕ. As opposed to the Fokker-Planck equations for Brownian motion, Eq. (5.23) contains an additional operator L_a, which does not commute with L_i, $[L_a, L_i] \neq 0$. This causes complications in the straightforward application of the path integral. However, some manipulation allows us to recover the free Brownian motion path integral, from which an exact reduction is possible.

Using $\mathcal{U} = \prod_k e^{\eta_k^2/4M_k}$, we make a similarity transformation as before to obtain an imaginary time Schrödinger equation with Hamiltonian

$$H = -L_a + \sum_k \gamma_k C_k^\dagger C_k + \sum_k \frac{\partial}{\partial\phi_k}g_k(\phi)\sqrt{M_k}(C_k + C_k^\dagger), \tag{5.25a}$$

with

$$C_k^\dagger = \frac{\eta_k}{2\sqrt{M_k}} - \sqrt{M_k}\frac{\partial}{\partial\eta_k}, \tag{5.25b}$$

$$C_k = \frac{\eta_k}{2\sqrt{M_k}} + \sqrt{M_k}\frac{\partial}{\partial\eta_k}. \tag{5.25c}$$

The reduced evolution operator is given by $R_{\rm red}(t) = \langle 0|e^{-Ht}|0\rangle$, where $|0\rangle$ is now the ground state for the operators $C_k, C_k^\dagger$. Writing $H_0 = -L_a + \sum_k \gamma_k C_k^\dagger C_k$, we apply Eq. (4.10) to obtain

$$R_{\rm red}(t) = e^{L_a t}\left\langle \exp\left(\int_0^t dt' \sum_k J_k(t')(c_k + c_k^*)\right)\right\rangle, \tag{5.26a}$$

$$J_k(t) = e^{-L_a t}\frac{\partial}{\partial\phi_k}g_k(\phi)\sqrt{M_k}e^{L_a t}. \tag{5.26b}$$

The functional average is equivalent to that for free Brownian motion and may be evaluated exactly to give

$$R_{\rm red}(t) = e^{L_a t}T\exp\left(\int_0^t dt_1\int_0^t dt_2 \times \sum_k J_k(t_1)S_k(t_1-t_2)J_k(t_2)\right), \tag{5.27a}$$

$$S_k(t_1 - t_2) = \tfrac{1}{2}e^{-\gamma_k|t_1-t_2|}, \tag{5.27b}$$

whereupon taking the time derivative of $R_{\rm red}(t)$,

$$L_{\rm eff}(t) = L_a + 2\sum_k J_k\int_0^t d\tau S_k(t-\tau)J_k(\tau - t), \tag{5.28}$$

with $J_k = J_k(0)$.

Equation (5.28) may be integrated using the identity

$$J_k(t) = e^{-L_a^\times t}J_k = \sum_{n=0}^\infty \frac{(-t)^n}{n!}(L_a^\times)^n J_k, \tag{5.29}$$

where $L_a^\times J_k = [L_a, J_k]$. The integration results in the exact formal expression

$$L_{\rm eff}(t) = L_a + \sum_k J_k \frac{1 - e^{-\gamma_k t}e^{L_a^\times t}}{\gamma_k - L_a^\times}J_k. \tag{5.30}$$

This may be written in the alternative form

$$L_{\rm eff}(t) = L_a + \sum_k\sum_{n=1}^\infty \frac{1}{\gamma_k^n}J_k(L_a^\times)^{n-1}J_k P(n, z_k), \tag{5.31a}$$

$$P(n,z) = \frac{1}{(n+1)!}\int_0^z ds\, s^{n-1}e^{-s} = 1 - e^{-z}\sum_{r=0}^{n-1}\frac{z^r}{r!}, \tag{5.31b}$$

with $z_k = \gamma_k t$ and $P(n,z)$ being the incomplete gamma function. When $t \gg 1/\gamma_k$, the above expressions simplify to

$$L_{\rm red} = L_a + \sum_k M_k \frac{\partial}{\partial\phi_k}g_k(\phi)(\gamma_k - L_a^\times)^{-1}\frac{\partial}{\partial\phi_k}g_k(\phi). \tag{5.32}$$

Reduced operators similar to this have been derived before for the colored noise problem but by different means [31,32]. In particular Grigolini [31] shows that different

approximations to Eq. (5.32) yield various results that have appeared in the literature.

The main problem in the study of systems with colored noise is to find a closed algebraic expression or approximation for the operators $L_{\rm eff}(t)$ or $L_{\rm red}$ above. This type of problem has attracted much attention in the literature and a variety of methods have been applied [15,33–35], but as stated before our objectives have been more modest and we will limit our investigation to just a few special cases. Now, the γ_k dependence of $P(n,z)$ prevents Eq. (5.31a) from being an explicit expansion of $L_{\rm eff}(t)$ in inverse powers of γ_k. However, $L_{\rm red}$ can be rendered into such a form, as is easily verified, and the lowest terms of this give the lowest terms of the "τ-expansion" result [36]:

$$L_{\rm red} = L_a + \sum_k \frac{M_k}{\gamma_k}\frac{\partial}{\partial\phi_k} g_k(\phi) \times \frac{\partial}{\partial\phi_k}\left\{ g_k(\phi) + \frac{1}{\gamma_k}\left(g_k \frac{\partial W_k}{\partial\phi_k} - \frac{\partial g_k}{\partial\phi_k} W_k \right) \right\}. \tag{5.33}$$

As further special examples, we note that for linear friction $W_k(\phi) = -\lambda_k\phi_k$ an explicit reduction is possible for $g_k(\phi) = 1$ and for $g_k(\phi) = \phi_k$. The latter case $g_k(\phi) = \phi_k$ is trivial. We have

$$L_a = -\sum_k \lambda_k \frac{\partial}{\partial\phi_k}\phi_k, \qquad J_k = \sqrt{M_k}\frac{\partial}{\partial\phi_k}\phi_k. \tag{5.34}$$

Now L_a and J_k commute, so $L_a^\times J_k = 0$ and Eq. (5.30) immediately yields

$$L_{\rm eff}(t) = L_a + \sum_k \frac{M_k}{\gamma_k}(1 - e^{-\gamma_k t})\frac{\partial}{\partial\phi_k}\phi_k\frac{\partial}{\partial\phi_k}\phi_k. \tag{5.35}$$

For $t \gg 1/\gamma_k$ the Stratonovich form of the Fokker-Planck equation is recovered. The result for $g_k(\phi) = 1$ is obtained most easily by use of Lie algebra methods [37]. Again L_a is given as above. We want to find a closed algebraic expression for

$$J_k(s) = e^{-L_a s}\sqrt{M_k}\frac{\partial}{\partial\phi_k}e^{L_a s}. \tag{5.36}$$

We note that the operators L_a and J_k are elements of a closed Lie algebra. Then since $J_k(s)$ satisfies the differential (Lie) equation

$$\frac{d}{ds}J_k(s) = [J_k(s), L_a], \tag{5.37}$$

use of the trial solution

$$J_k(s) = a_k(s)\phi_k + b_k(s)\frac{\partial}{\partial\phi_k}, \tag{5.38}$$

with the initial condition $J_k(0) = J_k$, where $a_k(s)$ and $b_k(s)$ are to be determined, readily gives

$$J_k(s) = e^{\lambda_k s}\sqrt{M_k}\frac{\partial}{\partial\phi_k}. \tag{5.39}$$

Inserting this into Eq. (5.28) and performing the τ integration yields

$$L_{\rm eff}(t) = \sum_k \frac{M_k}{\lambda_k + \gamma_k}\left(1 - e^{-(\lambda_k+\gamma_k)t}\right)\frac{\partial^2}{\partial\phi_k^2}. \tag{5.40}$$

Allowing for differences in notation, Eqs. (5.35) and (5.40) are trivial generalizations of known results [15].

VI. SUMMARY

In this paper we have studied the reduction of Fokker-Planck processes in which the correlation times of the dynamical variables can be separated into two distinct domains. This separation of time scales permits a decoupling of the full Fokker-Planck equation into two evolution equations for the fast and the slow variables. On long time scales, the distribution of the fast variables is determined essentially by a steady-state equation independent of the slow variables. The evolution of the slow variables is then determined by a projection of the original problem onto an equivalent problem in which the fast variables are maintained in that steady-state distribution. The fast variables thus act as an effective heat-bath, imposing their influence on the subsystem of slow variables without in turn being affected by them. On the other hand, the nonequilibrium distribution of the slow variables relaxes towards a state determined by the coupling of the slow variables to the fast.

A number of methods for achieving the decoupling are known. Our main result is that the projection process used to obtain the dynamics of the reduced problem is equivalent to performing a functional integration over the fast variables. In particular we have shown that this procedure is very conveniently formulated in the coherent state representation of the path integral. The reduced evolution operator is then found by performing the relevant functional integral. An exact evaluation of the path integral is only possible for a few processes. The cases of free Brownian motion and a system with exponentially correlated colored noise were studied as specific examples where this is possible. For Brownian motion in a potential and indeed for general stochastic processes, an exact evalution is not possible. In this case it was shown that the path integral may be expanded in an ordered cumulant expansion, permitting an approximation to the reduced evolution operator that is good provided the relaxation times of the fast variables are sufficiently rapid.

In the applications studied, no specific computational gains were apparent through the introduction of the path integral representation. The analysis could equally well have been performed in operator form (see Sec. III) and identical results for the reduced evolution operator would have been obtained. In this sense we are in a situation like that of quantum mechanics, where canonical quantization and path integral quantization are known to be equivalent appoaches to the same problem. However, in quantum mechanics it is known that path integral methods admit manipulations of the theory that may be more cumbersome using only operator methods. These include series rearrangements and asymptotic expansions, among others [38]. Possible applications of such techniques to the reduction problem have not been investigated here.

Thus the gain achieved has been for the present a purely technical one. Functional integration is a very elegant form of expression for reduction (heat bath) problems and we have shown exactly how overdamped stochastic processes may be described in this approach.

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